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LETTER TO THE EDITOR

Dimensional crossover in directed percolation†

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Abstract. We study the dimensional crossover in directed percolation in three dimensions. Bonds are allowed to have different concentrations along the three cartesian axes of the lattice. Through a position space renormalisation group we obtain the phase diagram where nonpercolating, 1D, 2D and 3D percolating phases are present. We find that, contrary to what happens in undirected percolation, the isotropic fixed points are unstable with respect to concentration anisotropy. Numerical estimates are given for the values of critical probabilities and exponents, which are in fairly good agreement with other results, where available.

The isotropic flow of a fluid through a porous medium is a percolation process (for a review see e.g. Essam 1980). The introduction of a global external bias, such as the gravitational field, changes a number of essential features of the problem, the resulting process being called directed percolation (for a review see Kinzel (1983) and references therein). Directed percolation has been shown to be related to Reggeon field theory (Cardy and Sugar 1980), which in turn is connected to Markov processes in chemistry and biology (Grassberger and de la Torre 1979).

Within the context of the modern theory of critical phenomena, the interest of directed percolation problems stems mainly from the fact that they exhibit critical properties quite distinct from those of ordinary percolation. Indeed, critical exponents are different in either case (Blease 1977a, b, c), and the upper critical dimensionality is five (Obukhov 1980), whereas it is six for ordinary percolation. Other features of directed percolation find no parallel in ordinary percolation. Due to the directional bias, the percolation cluster is extremely anisotropic in shape, and can be thought of as a cone whose axis is parallel to the 'easy' direction. One can then define a correlation length ξ_ϕ along a direction at an angle ϕ from the 'easy' direction, which diverges as

$$\xi_\phi \sim (p_c(\phi) - p)^{-\nu_\parallel(\phi)} \quad (1)$$

where $p_c(\phi)$ is the critical bond (or site) concentration threshold for percolation along that direction. In two dimensions, Domany and Kinzel (1981) have shown that $\nu_\parallel(\phi)$ can only have two values, corresponding respectively to $\phi = 0$ and $\phi \neq 0$. One can also define a pseudo-correlation length $\xi_{\perp,\phi}$ (Klein and Kinzel 1981), which measures the spread of the percolating cone perpendicular to the direction of observation ϕ , and whose divergence is related to an exponent $\nu_\perp(\phi)$, similarly to (1) above.

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The purpose of the present work is to investigate the dimensional crossovers in directed bond percolation on a simple cubic lattice, by allowing the bond concentrations along the three cartesian axes to vary independently. Concentration anisotropy is irrelevant, in a renormalisation-group sense, in undirected percolation (see e.g. Oliveira (1982) and references therein); however, it has been shown to be relevant in two-dimensional directed percolation (Domany and Kinzel 1981, Oliveira 1983). Thus, it is interesting to extend the discussion of concentration anisotropy effects to a three-dimensional directed percolation problem, in order to check how it differs from its undirected counterpart. By means of a position-space renormalisation group (Young and Stinchcombe 1975) with cells suitably chosen to deal with concentration anisotropy in the context of a directed-lattice problem (Oliveira 1983) we obtain the phase diagram where one-, two- and three-dimensional percolating phases occur. Apart from extracting quantitative information about the different dimensional crossovers, we can analyse the behaviour along different directions of observation. Indeed, Domany and Kinzel (1981) proposed that the effect of varying the angle of observation ϕ could be obtained by keeping the direction of observation fixed and allowing anisotropic bond concentrations in such a way that, on a square lattice $\phi = \tan^{-1}(y/x - \pi/4)$, where x and y are the bond concentrations along the horizontal and vertical directions, respectively. In what follows, we first discuss the renormalisation group transformation; the results for fixed points, critical exponents and the phase diagram are then presented and discussed; finally we summarise our findings and present our conclusions.

Consider a simple cubic lattice, and let x , y and z denote the concentration of directed bonds along each of the cartesian directions. In order to describe anisotropy properties within a position space renormalisation group framework, the transformation of x , y and z under a length scaling (by a factor b) must have the form

$$x' = R_b(x, y, z), \quad y' = R_b(y, z, x), \quad z' = R_b(z, x, y) \quad (2)$$

so that no other asymmetry is introduced in the problem.

In this work we extend to three dimensions the prescription introduced by Oliveira (1983) which preserves the symmetry properties (2). In figure 1(a) we show a cell in the original lattice, which is renormalised into the bonds of figure 1(b); the scaling factor is $b=2$. The probability x' of having an x -bond in the renormalised cell is obtained by counting all possible configurations that start at the origin 0 and get across the original cell along the x direction; similarly for y' and z' .

The process of counting cell configurations can be greatly simplified with the use of the 'break-collapse' method (Tsallis and Levy 1981), properly modified to keep track of the direction of the collapsed bond (Tsallis and Redner 1983).

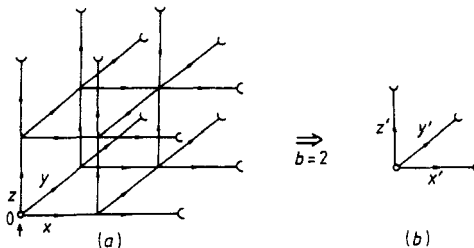


Figure 1. Cells used in the renormalisation group. The cell in (a) is renormalised into (b) by counting configurations that start at 0 and leave the cell along the different directions.

Before we present our results we should comment on two aspects of the cell-RG approach to directed lattice problems. First, the correlation length exponents obtained from this approach are estimates for ν_{\parallel} (de Queiroz (1983) and references therein). Second, we use cubic cells of linear size b (in the present case we set $b=2$), which implies that ν_{\parallel} is always measured along the [111] direction. The angular position of the 'easy' direction with respect to the [111] direction, in turn, is determined by the relative concentrations (Domany and Kinzel 1981, Oliveira 1983): in spherical coordinates we have $\phi = \tan^{-1} y/x - \pi/4$ and $\theta = \tan^{-1} [(x^2 + y^2)^{1/2}/z - \frac{1}{4}\pi]$. Solving recursion relation (2) for non-trivial fixed points

$$x^* = R_2(x^*, y^*, z^*), \quad y^* = R_2(y^*, z^*, x^*), \quad z^* = R_2(z^*, x^*, y^*) \quad (3)$$

we obtain the results displayed in table 1. As usual (Wilson and Kogut 1974), critical exponents follow from the relevant eigenvalues ($\lambda_i > 1$) of the linearised recursion relations around the fixed points through $\nu_i = \ln b / \ln \lambda_i$; the correlation-length exponent at each fixed point is given by the eigenvalue associated to the eigenvector that points away from the critical surface and is denoted by ν_1 in table 1. Wherever there is more than one relevant eigenvalue, we display the corresponding exponent ν_i ($i > 1$) and the crossover exponent $\phi = \nu_1 / \nu_i$. Physically, the exponents ν_i ($i > 1$) describe the divergence of the correlation length as the critical point is approached tangentially to the critical surface.

Iterating recursion relations (2) we obtain the phase boundaries as the RG trajectories linking pairs of non-trivial fixed points. Before presenting the full phase diagram, it is instructive to discuss some special cases.

Setting $z=0$ we recover the two-dimensional results of Oliveira (1983), reproduced in figure 2(a): a boundary BAB separates percolating and non-percolating phases. Unlike ordinary percolation, concentration anisotropy drives the system to a new critical behaviour, which is that of fixed point B. This is consistent with the idea that anisotropy is equivalent to rotating the percolating cone, for a fixed direction of observation, thus giving rise to a new exponent ν_{\parallel} . Further, the fact that at A the boundary is curved towards the full-lattice point (1, 1) is a reminder that the *location* of bonds with respect to the origin (as well as their amount) is crucial to the formation of a percolating cone (Domany and Kinzel 1981); recall that for anisotropic undirected percolation the critical boundary is the straight line $x+y=1$ (Sykes and Essam 1963).

Setting, e.g., $x=1$, we obtain the flow diagram of figure 2(b). This region corresponds to having all x -bonds present, so 0 is a one-dimensional fixed point, to which the flow lines below the curve BCB converge. If we increase the concentrations of y - and z -bonds always keeping $y=z$, we end up with a three-dimensional percolating phase as point B is crossed. On the other hand, if $y \neq z$ the system orders into a two-dimensional percolating phase before it reaches the three-dimensional phase as y and z are increased. At C the eigendirections are t and u , defined by the lines $z+y=0.560$ and $z-y=0$, respectively; the crossover exponent ϕ (defined as $t \sim u^{\phi}$) being smaller than one means that, close enough to C, any small additional amount of y and z bonds in such a way that $z+y > 0.560$ will drive the system into a three-dimensional percolating phase. In the same way, if $z+y < 0.560$, the system will percolate in only one dimension. Also, the fact that point D in figure 2(b) is located at $z \neq 0$ is an evidence that not only the amount, but also the location of z -bonds is crucial to the formation of a 3D percolating phase along the [111] direction. In analogy with the two-dimensional case of Domany and Kinzel (1981), the critical probability

Table 1. Fixed points, eigenvalues and critical exponents obtained from our RG transformations. Equivalent fixed points are generated through cyclic permutations of x, y and z .

| | Fixed points | | | Eigenvalues | | | Exponents | Remarks |
|---|--------------|-------|-------|-------------|-------------|-------------|---|--|
| | x^* | y^* | z^* | λ_1 | λ_2 | λ_3 | | |
| A | 0.671 | 0.671 | 0 | 1.624 | 1.099 | 0 | $\nu_1 = 1.43, \nu_2 = 7.34$ $\phi = 0.195$ | 2D isotropic fixed point; $x^* = y^* = 0.6445 \pm 0.0005$ (a), $\nu_{ } = 1.734 \pm 0.002$ (b) |
| B | 1 | 1/2 | 0 | 3/2 | 3/4 | 0 | $\nu_1 = 1.71$ | 2D anisotropic fixed point; $x^* = 1, y^* = \frac{1}{2}, \nu = 2$ (c) |
| C | 1 | 0.280 | 0.280 | 1.869 | 1.331 | 0.788 | $\nu_1 = 1.11, \nu_2 = 2.42$ $\phi = 0.457$ | 3D anisotropic (with 2D isotropy) fixed point |
| D | 1 | 1 | 0.146 | 1.717 | 0 | 0 | $\nu_1 = 1.28$ | 3D anisotropic fixed point |
| E | 0.421 | 0.421 | 0.421 | 2.056 | 1.164 | 1.164 | $\nu_1 = 0.962, \nu_2 = 4.56$ $\phi = 0.211$ | 3D isotropic fixed point $x^* = y^* = z^* = 0.382 \pm 0.001$ (d), $\nu_{ } = 1.27 \pm 0.01$ (b) |

(a) Blease (1977a, c), (b) Kinzel (1983), (c) Domany and Kinzel (1981), (d) De'Bell and Essam (1983).

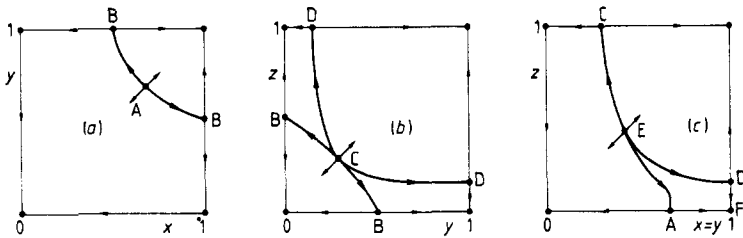


Figure 2. (a), $z=0$ section of the full flow diagram; the critical curve separates two-dimensional percolating and non-percolating phases. (b), $x=1$ section of the full flow diagram; the critical curves separate the different percolating phases and the non-percolating phase. (c), Projection on yz plane of the $x=y$ section of the full phase diagram; the critical curves separate non-percolating, two- and three-dimensional percolating phases. Similar diagrams occur with cyclic permutations of x , y and z .

z_c varies from zero (along directions close enough to the xy plane in the first octant) to one (along the $[001]$ direction); we have not tried to work out an exact expression for $z_c(\theta, \phi)$ which would be the three-dimensional version of $p_c(\alpha)$ of Domany and Kinzel (1981).

In figure 2(c) we show the flow diagram corresponding to two-dimensional concentrational isotropy. Below the curve CEA lies a non-percolating phase, a two-dimensional percolating phase exists within the region $EAFD$, and the three-dimensional phase is above CED . Note that as the concentrations increase with $z > x = y$, the transition is from a non-percolating to a three-dimensional percolating phase. This means that, although the system does not percolate in each xy plane, the lack of bonds in these planes is compensated by the z -bonds; that is, the percolating path jumps to the next layer. On the other hand, if $z < x = y$, there is an intermediate two-dimensional phase, as the concentrations increase.

At the isotropic fixed point (E in figure 2(c)), all three eigenvalues are relevant, two of them degenerate. In three-dimensional probability space, the eigenvector associated to the largest eigenvalue points along the $[111]$ direction; the eigenvectors associated to the degenerate (smaller) eigenvalues span the plane perpendicular to the $[111]$ direction passing through E. Once again, the fact that E is a completely unstable fixed point shows that the slightest degree of concentration anisotropy will drive the system towards anisotropic behaviour. As can be seen from figure 3 (where equivalent points are denoted by the same letters), points B, D and F are stable, respectively within each of the critical surfaces that meet tangentially along the four EC lines (B and D) and in the region between those surfaces (F). As A and C have two-dimensional isotropy, they are stable only along EA and EC, respectively.

Figure 3 shows an overall view of the phase diagram; note that one-dimensional percolating phases exist only when either x or y or z is 1.

In summary, we have obtained the phase diagram for the anisotropic three-dimensional directed percolation problem on a simple cubic lattice, by means of a position-space renormalisation group. Our results for critical probabilities and exponents are in fairly good numerical agreement with other estimates, where these are available (see table 1). We have found regions of non-percolating and one-, two- and three-dimensional percolating phases, although the one-dimensional phases only occur for either x or y or $z = 1$. The crossovers between the different phases have been studied, with the general conclusion that isotropic fixed points are unstable with respect

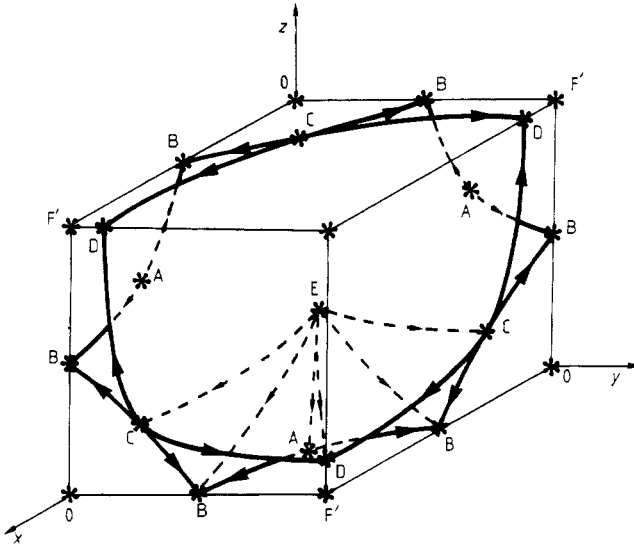


Figure 3. The full phase diagram (schematic). Equivalent fixed points are denoted by the same letters. Also shown are typical flow lines leaving the three-dimensional isotropic fixed point E.

to concentration anisotropy, contrary to what happens in ordinary (undirected) percolation. Thus, if one observes isotropic directed percolation along a direction other than $[111]$, the critical behaviour will be described by a different set of exponents.

As a final remark, we wish to point out that the cells we used do not properly describe anisotropy effects in undirected percolation; conversely, the cells that properly describe these effects in undirected percolation (Oliveira 1982) do not reproduce the behaviour we found. This question certainly deserves careful discussion before we try to investigate the anisotropic diode-resistor crossover. Note that this problem does not arise in the discussion of the crossover between *concentrationaly isotropic* diodes and resistors (Redner and Brown 1981) because there one is confined to a subspace of parameter space, within which results yielded by cells similar to ours are reliable (Oliveira 1982).

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